Deriving the the Euler Product Formula from the Riemann zeta function, and a brief example.

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Number Theory

Abstract

Swiss mathematician Leonhard Euler made numerous discoveries in the field of analytic number theory. In his 1737 thesis, Euler introduced what would come to be known today as the Euler Product Formula. He would go on to prove that this product was in fact a formula for the Riemann zeta function. Arguably one of the hottest topics in the field of modern number theory is the Riemann zeta function. Though the zeta function was first known to Euler, it was his German neighbor to the north Bernhard Riemann who studied this function at lengths. It was in his 1859 paper that Riemann introduced an explicit formula for the number of primes up to a pre-determined limit. The caveat of this function being that it depended on knowing the values for which the zeta function was zero. Riemann conjectured that all (non-trivial) zeroes lived on the critical strip at $x = \frac{1}{2}$. Still an open problem today and one who's solution comes with a large reward. Our focus will be on deriving Euler's Product Formula from the Riemann zeta function. The paper will conclude with a brief example.

1 Introduction

In the early 1300's, a mathematician by the name of Nicolas Oresema showed that the harmonic series used today is in fact divergent. Euler knew this and wondered if the harmonic series involving only primes $(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...)$ was also divergent. He began studying powers of these primes and would soon formulate his Product Formula. Much of anayltic number theory relies on one beautiful formula:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{primes\, p} (1 - \frac{1}{p^s})^{-1}$$

Now that we have our formula, how is it that Euler came up with these ideas? It was clear to him that the classic harmonic series diverged, but could this new harmonic series composed only of primes also diverge? Euler had an incredibly intutive idea about how he would prove whether this infinte sum had a finite answer or not. He would break up the classic harmonic function into two seperate summations. The first summation made up of the recipricols of all composite numbers and the second summation being the recipricols of all primes, i.e.,

$$[\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + ...] + [1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...]$$

The goal was to show that the first summation had a finite answer and the cause of the divergence was laced within the second summation. Though this problem turned out not to be so simple. Consider the Grandi series(named after the Italian mathematician Guido Grandi):

$$S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - .... = \sum_{n=0}^{\infty} (-1)^n$$

Then we can group this series in two different ways, i.e.,

$$S = (1 - 1) + (1 - 1) + (1 - 1) + (1 - 1) + ... \text{ or}$$

$$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + (-1 + 1) + ...$$

Then it is clear that the first summation should equal to 0, while the second summation should be 1. This is a classic issue that arises when trying to break up infinite sums. This would prove to be a hurdle in his effort to solve this problem, but this is Euler we are talking about, so of course he found a newfangled way to solve this problem.

2 The zeta function

With this new harmonic series composed only of primes in mind, Euler decided to look at a new sum that was closely related. He began by taking a a complex number s such that $Re(s)$ was just slightly larger than 1. He then took the old harmonic series and made a new function using this s-value, namely:

$[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...]\Rightarrow[1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+...]$

which would come to be known as the famous Riemann zeta function $\zeta(s)$.When $Re(s)>1$, Euler knew that this series converged (today we can prove convergence by the p-series test), so it was now possible to break up this series into two infinite pieces, i.e.,

$\zeta(s)=[\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{9^s}+...]+[1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+...]$

His new idea was to show as $Re(s)\to 1$, the summation on the right was in fact divergent. In this proof, Euler would come to discover his Product Formula, and now we will derive his Product Formula from the zeta function.

3 Deriving the Euler Product Formula from the zeta function

This proof will use a modifed sieve method. Let $Re(s)>1$ and consider the Riemann zeta function $\zeta(s)=\sum\limits_{n=1}^{\infty}\frac{1}{n^s}$.

Then $\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+...$ (1)

Multiply both sides by $\frac{1}{2^s}$ to get,

$\frac{1}{2^s}\zeta(s)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+...$ (2)

We can now subtract this new equation from the original to get,

$\zeta(s)-\frac{1}{2^s}\zeta(s)=(1-\frac{1}{2^s})\zeta(s)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+...$ (1) $-$ (2)

So we have removed all multiples of two from our original equation.

We can take our new equation and multiply by a factor of $\frac{1}{3^s}$ to get,

$\frac{1}{3^s}(1-\frac{1}{2^s})\zeta(s)=\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+...$ (3)

Again if we subtract this equation from the original zeta function we get,

$\zeta(s)-\frac{1}{3^s}(1-\frac{1}{2^s})\zeta(s)=(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=(1)-(3)$

$=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+...$

Thus we have removed all factors of both two and three from our original equation. We can repeat this process again, multiplying this new equation by $\frac{1}{5^s}$ on both sides to get,

$\frac{1}{5^s}(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=\frac{1}{5^s}+\frac{1}{25^s}+\frac{1}{35^s}+\frac{1}{55^s}+\frac{1}{65^s}+...$ (4)

Subtract this new equation from the original zeta function to get,

$\zeta(s)-\frac{1}{5^s}(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=(1)-(4)$

$=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+...$

So we have removed all factors of two, three, and five. Now it should be clear that we can repeat this process iteratively for $\frac{1}{p^s}$ where p is prime. The result is...

$...(1-\frac{1}{17^s})(1-\frac{1}{13^s})(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\zeta(s)=1$

Dividing both sides by $\zeta(s)$ results in,

$...(1-\frac{1}{17^s})(1-\frac{1}{13^s})(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})=\frac{1}{\zeta(s)}$

$\Rightarrow\zeta(s)=\frac{1}{(1-\frac{1}{2^s})(1-\frac{1}{3^s})(1-\frac{1}{5^s})(1-\frac{1}{7^s})(1-\frac{1}{11^s})(1-\frac{1}{13^s})(1-\frac{1}{17^s})...}=\prod_{primes\, p}(1-\frac{1}{p^s})^{-1}$

$\square$

4 An example

Let $s=2$ and consider $\zeta(s)$. Then,

$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+...$

Goal is to show $\zeta(2)=\frac{\pi^2}{6}$.

Proof (Euler 1735):

Consider the Maclaurin series for $sin(\pi x)$.

$sin(\pi x)=\pi x-\frac{(\pi x)^3}{3!}+\frac{(\pi x)^5}{5!}-\frac{(\pi x)^7}{7!}+...=\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\pi^{2n+1}x^{2n+1}}{(2n+1)!}$ (1)

It is important to note here that the coefficient of x^3 is $\frac{-\pi^3}{6}$. Recall that the Fundamental Theorem of Algebra tells us that we can always factor a finite polynomial into a linear product of its roots. But can we do this for infinite polynomials? Note that the complex-valued function $sin(\pi x)$ is entire, in other words, our function is holomorphic at all finite points in the whole complex plane. This is important because the Weierstrass factorization theorem says that entire functions can be represented by a linear product involving their zeroes(even infinite polynomials). Let us factor our expansion for $sin(\pi x)$ using the zeroes of $sin(\pi x)$ to get,

$sin(\pi x) = \pi x(1-x)(1+x)(1-\frac{x}{2})(1+\frac{x}{2})(1-\frac{x}{3})(1+\frac{x}{3})(1-\frac{x}{4})(1+\frac{x}{4})...$

Recall the difference of squares formula that says $(a-b)(a+b) = a^2 - b^2$. So we can rewrite our factorization of $sin(\pi x)$ as,

$sin(\pi x) = \pi x(1-x^2)(1-\frac{x^2}{4})(1-\frac{x^2}{9})(1-\frac{x^2}{16})...$ (2)

Using our new factorization (2), if we want to get an x^3 term, we must multiply the initial πx by one single term in the factorization that is of the form $\frac{-x^2}{n^2}$. After multiplying each factor by x, we can pull out a x^3 from each term to get,

$sin(\pi x) = \pi x - \pi x^3(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...) + ...$ (3)

Note that the rest of the expansion is irrelevant as we now have a new expression for our x^3 coefficient. Recall that in our original expansion (1) the coefficient for $x^3$is $\frac{-\pi^3}{6}$. But in our new expansion (3), we have that the coefficient is $-\pi(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...)$. Recall that $\zeta(2) = (1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...)$. So the coefficient for x^3 can be written as $-\pi\zeta(2)$. Now we can set the two coefficients equal to each other and solve for $\zeta(2)$.

We have $-\pi\zeta(2) = \frac{-\pi^3}{6}$.

$\Rightarrow\zeta(2) = \frac{-\pi^3}{6} \cdot \frac{-1}{\pi} = \frac{\pi^2}{6}$.

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