Arithmetic functions and their Dirichlet series

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Abstract

Given an arithmetic function, one can establish its Dirichlet series. In this paper we will examine four familiar arithmetic functions and their corresponding Dirichlet series. The aim is to represent these series as products and ratios of the Riemann zeta function.

Arithmetic Functions

Definition 1.1 A function $f : \mathbb{N} \to \mathbb{C}$ is called an *arithmetic function.* An arithmetic function f is called *multiplicative* if it satisfies the relation:

$$f(n_1 n_2) = f(n_1) f(n_2) \ \forall n_1, n_2 \in \mathbb{N} \colon (n_1, n_2) = 1$$

If the relation holds for all $n_1, n_2 \in \mathbb{N}$ (without the restriction $(n_1, n_2) = 1$), then f is called *completely multiplicative*.

Examples

1) The *Möbius function* $\mu : \mathbb{N} \to \mathbb{C}$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is not square free} \\ (-1)^r & \text{if } n = p_1 p_2 \ldots p_r \end{cases}$$

2) The *divisor function* $\tau : \mathbb{N} \to \mathbb{C}$ is the number of divisors of $n \in \mathbb{N}$, defined by

$$\tau(n) = \sum_{d|n} 1$$

3) The sum of divisors function $\sigma \colon \mathbb{N} \to \mathbb{C}$ is the sum of all the divisors of $n \in \mathbb{N}$, defined by

$$\sigma(n) = \sum_{d|n} d$$

4) The Euler totient function $\phi \colon \mathbb{N} \to \mathbb{C}$ is the number of integers $\leq n$ that are co-prime to n, given by

$$\phi(n) = \# \{1 \leq m \leq n : (m, n) = 1.\}$$

Note that each function is multiplicative, but none are completely multiplicative.

Definition 1.2 Given an arithmetic function $f(n)$, the series

$$D_f(n) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{f(1)}{1^s} + \frac{f(2)}{2^s} + \frac{f(3)}{3^s} + \ldots$$

is called the *Dirichlet series* associated with f. Consider the arithmetic function $f(n) = 1$. Then its Dirichlet series is,

$$D_1(n) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

Definition 1.3 Given two arithmetic functions f and g, the *Dirichlet convolution* $f \star g$ is the arithmetic function defined by

$$(f \star g)(n) = \sum_{d|n} f(d) g(\tfrac{n}{d}) \quad (n \in \mathbb{N})$$

Note also that $D_{f \star g} = D_f D_g$

Dirichlet Series

The Möbius function Recall the Möbius function $\mu(n)$ from Example 1. Then its Dirichlet series is,

$$D_\mu(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

Consider the Euler product

$$F = \prod_{k=1}^{\infty} (1 - \frac{1}{p_k^s}) = (1 - \frac{1}{p_1^s})(1 - \frac{1}{p_2^s})(1 - \frac{1}{p_3^s})(1 - \frac{1}{p_4^s})(1 - \frac{1}{p_5^s})(1 - \frac{1}{p_6^s}) \ldots$$

If we continuously multiply out the terms of F, we will get

$$F = (1 - \frac{1}{p_1^s})(1 - \frac{1}{p_2^s})(1 - \frac{1}{p_3^s})(1 - \frac{1}{p_4^s})(1 - \frac{1}{p_5^s})(1 - \frac{1}{p_6^s}) \ldots$$

$$= (1 - \frac{1}{p_1^s} - \frac{1}{p_2^s} + \frac{1}{p_1^s p_2^s})(1 - \frac{1}{p_3^s})(1 - \frac{1}{p_4^s})(1 - \frac{1}{p_5^s})(1 - \frac{1}{p_6^s}) \ldots$$

$$= (1 - \frac{1}{p_1^s} - \frac{1}{p_2^s} - \frac{1}{p_3^s} + \frac{1}{p_1^s p_2^s} + \frac{1}{p_1^s p_2^s} + \frac{1}{p_2^s p_3^s} - \frac{1}{p_1^s p_2^s p_3^s})(1 - \frac{1}{p_4^s})(1 - \frac{1}{p_5^s})(1 - \frac{1}{p_6^s}) \ldots$$

$$\vdots$$

$$= 1 - (\frac{1}{p_1^s} + \frac{1}{p_2^s} + \ldots) + (\frac{1}{p_1^s p_2^s} + \ldots + \frac{1}{p_1^s p_3^s} + \frac{1}{p_2^s p_3^s} + \ldots) - \ldots$$

$$= 1 - \sum_{0<i} \frac{1}{p_i^s} + \sum_{0<i<j} \frac{1}{p_i^s p_j^s} - \sum_{0<i<j<k} \frac{1}{p_i^s p_j^s p_k^s} + \sum_{0<i<j<k<l} \frac{1}{p_i^s p_j^s p_k^s p_l^s} - \ldots$$

Given any summation $\sum \frac{\pm 1}{p_1 p_2 \ldots p_r}$, when r is odd, the numerator is -1, whereas when r is even, the numerator is $+1$. Note also that there are no terms that contain a squared prime, thus we really have

$$D_\mu(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = F = \prod_{k=1}^{\infty} (1 - \frac{1}{p_k^s}) = \zeta^{-1}(s)$$

Thus the Dirichlet series of the möbius function is just the reciprocal of the Riemann zeta function.

The divisor function Recall the divisor function $\tau(n)$ from Example 2. Consider its Dirichlet series,

$$D_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

Next consider the Dirichlet product where $f(n) = 1$ and $g(n) = 1$, namely

$$f \star g(n) = 1 \star 1(n) = \sum_{d|n} 1(d)1(\tfrac{n}{d}) = \sum_{d|n} 1 = \tau(n)$$

thus $\tau(n) = 1 \star 1(n)$. So to calculate the Dirichlet series of the divisor function, we can instead consider the new Dirichlet series,

$$D_\tau(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \sum_{k=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \cdot \zeta(s) = \zeta^2(s)$$

Consider instead the Euler product of $D_\tau(s)$,

$$D_\tau(s) = \prod_p \left(1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{p^{2s}} + \ldots + \frac{\tau(p^k)}{p^{ks}} + \ldots\right)$$

$$= \prod_p \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \ldots + \frac{k+1}{p^{ks}} + \ldots\right)$$

$$= \prod_p \frac{1}{(1-p^{-s})^2} \quad \text{(Using the power series } (1-x)^2 = \sum_{k=0}^{\infty} (k+1)\, x^k) \Rightarrow$$

$$D_\tau(s) = \prod_p \frac{1}{(1-p^{-s})^2} = \zeta(s) \cdot \zeta(s) = \zeta^2(s)$$

So it is clear that the Dirichlet series $D_\tau(s)$ is $\zeta^2(s)$.

Sum-of-divisors function Consider the Dirichlet series of the function $\sigma(n)$ from Example 3.

$$D_\sigma(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s}$$

Consider the arithmetic functions $f(n) = n$ and $g(n) = 1$. Then

$$f \star g(n) = \sum_{d|n} f(d)\, g\left(\frac{n}{d}\right) = \sum_{d|n} d * 1 = \sum_{d|n} d = \sigma(n)$$

So $\sigma(n)$ is the Dirichlet product of the unit function $1(n)$, and the identity function $j(n)$, so using properties of the Dirichlet product we have,

$$D_\sigma(s) = D_1(s) \cdot D_j(s)$$

$$= \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{j(n)}{n^s}$$

$$= \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{n}{n^s}$$

$$= \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{1}{n^{s-1}}$$

$$= \zeta(s) \cdot \zeta(s-1)$$

Thus the Dirichlet series for $\sigma(n)$ is $\zeta(s) \cdot \zeta(s-1)$.

Euler's totient function Consider the Dirichlet series of the function $\phi(n)$ from Example 4.

$$D_\phi(s) = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

Then we can derive the Dirichlet series using the following Euler product,

$$D_\phi(s) = \prod_p \left(1 + \frac{\phi(p)}{p^s} + \frac{\phi(p^2)}{p^{2s}} + ... + \frac{\phi(p^k)}{p^{ks}} + ...\right)$$

$$= \prod_p \left(1 + \frac{p-1}{p^s} + \frac{p^2-p}{p^{2s}} + ... + \frac{p^k-p^{k-1}}{p^{ks}} + ...\right)$$

$$= \prod_p \left(1 + \frac{p-1}{p^s} \cdot \frac{1}{1-p^{1-s}}\right) \quad \text{(by summing the geometric progression)}$$

$$= \prod_p \left(\frac{1-p^{-s}}{1-p^{-(s-1)}}\right)$$

$$= \frac{\zeta(s-1)}{\zeta(s)}$$

So the Dirichlet series for Euler's totient function $\phi(n)$ is $\frac{\zeta(s-1)}{\zeta(s)}$.