# Topological Conjugacy and Applications to Dynamical Systems

Storm Frazier

December 17, 2019

**Abstract**

A focal point of Dynamical Systems is the study of topological conjugacies between functions. The reason that a topological conjugacy is important is if the dynamics of one function can be solved, then the dynamics of any topologically conjugate function follow immediately. We will begin by studying the topological conjugacy between the logistic map and the tent map. Then we will introduce the concept of a semi-conjugacy and show a few examples using MATLAB of how erratic these functions can get.

# 1 Topological Conjugacy

**Definition.**

- A function $f : X \mapsto Y$ is *injective* if:
  - $f(a) = f(b)$ then $a = b$
  - $f$ maps distinct elements to distinct elements
- A function $f : X \mapsto Y$ is *surjective* if:
  - For all $y \in Y$ there exists $x \in X$ such that $f(x) = y$
  - Every element in $Y$ has at least one element in $X$ that maps to it under $f$.

**Definition.** A function $\phi : X \mapsto Y$ is a *homeomorphism* if:

- $\phi$ is injective
- $\phi$ is surjective
- $\phi$ is continuous
- $\phi^{-1}$ is continuous

**Definition.** Let $F : X \mapsto X$ and $G : Y \mapsto Y$ be two functions. We say that $F$ and $G$ are *topologically conjugate* if there exists a homeomorphism $h : X \mapsto Y$ such that:

$$h \circ F = G \circ h$$

**Proposition 1** On the unit interval $[0, 1]$, the logistic map $G(x) = 4x(1 - x)$ is *topologically conjugate* to the tent map $T(x) = 1 - 2|x - \frac{1}{2}|$. Furthermore, the conjugacy is $h(x) = (1 - cos\pi x)/2$

**Proof:** To verify that $G$ and $T$ are topologically conjugate, it suffices to show that $h(T(x)) = G(h(x))$ for all $x \in [0, 1]$. We will first show that it works for $0 \leq x \leq \frac{1}{2}$ and then for $\frac{1}{2} \leq x \leq 1$. Consider $x \in [0, 1]$, then:

$$\begin{aligned} G(h(x)) &= 4h(x)(1 - h(x)) \\ &= 4(\frac{1 - cos\pi x}{2})(\frac{1 + cos\pi x}{2}) \\ &= 1 - cos^2\pi x \\ &= sin^2\pi x \end{aligned} \tag{1}$$

Now we can make use of the fact that $T(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$. So, for $x \in [0, \frac{1}{2}]$:

$$\begin{aligned} h(T(x)) &= \frac{1 - cos\pi T(x)}{2} \\ &= \frac{1 - cos2\pi x}{2} \\ &= sin^2\pi x \end{aligned} \tag{2}$$

Where the last equality holds by the double angle formula for cosine. Thus $h(x)$ does define a topological conjugacy on $0 \leq x \leq \frac{1}{2}$. Next we need to check for $\frac{1}{2} \leq x \leq 1$. We can make use of the fact that $T(x) = 2 - 2x$ for $\frac{1}{2} \leq x \leq 1$. So, for $x \in [\frac{1}{2}, 1]$:

$$\begin{aligned} h(T(x)) &= \frac{1 - cos\pi T(x)}{2} \\ &= \frac{1 - cos\pi(2 - 2x)}{2} \\ &= \frac{1 - cos(2\pi - 2\pi x)}{2} \\ &= \frac{1 - cos(-2\pi x)}{2} \\ &= \frac{1 - cos(2\pi x)}{2} \\ &= sin^2(\pi x) \end{aligned} \tag{3}$$

We have that $h(T(x)) = G(h(x))$ for all $x \in [0, 1]$, therefore $G$ and $T$ are topologically conjugate with conjugacy $h(x) = (1 - cos\pi x)/2$. This completes the proof.

**Definition.** Let $F : X \mapsto X$ and $G : Y \mapsto Y$ be two functions. We say that $F$ and $G$ are *topologically semi-conjugate* if there exists a surjection $h : X \mapsto Y$ such that:

$$h \circ F = G \circ h$$

Recall the the tent map,

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and consider it composed with itself to get,

$$T \circ T(x) = T_2(x) = \begin{cases} 4x & 0 \leq x \leq \frac{1}{4} \\ 2 - 4x & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 4x - 2 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 4 - 4x & \frac{3}{4} \leq x \leq 1 \end{cases}$$

We can keep composing the tent map with itself to generate new functions. These iterations of the tent map are strikingly similar to the *sawtooth wave*. The main goal of this next section will be to find a semi-conjugacy $h(x)$ from $T_2(x)$ to $T(x)$ and observe how chaotic that map looks when graphed. We will then look at more iterations of the tent map and see how chaotic the graphs of the semi-conjugacies from $T_n(x)$ to $T(x)$ can get. Because $h(x)$ is not going to be a smooth function, we will approximate the function numerically using MATLAB.

**Definition** We can describe the orbits of the tent map using their "left-right" **itineraries**: for each $c \in [0,1]$, we assign a sequence of L's and R's by writing 'L' if the $n$th point of the orbit of $c$ is in $[0, \frac{1}{2}]$ and 'R' if the $n$th point of the orbit of $c$ is in $(\frac{1}{2}, 1]$.

**Proposition 2** Given a sequence $X = (x_0\ x_1\ x_2\ x_3\ ...)$ of L's and R's there is a unique point $\alpha \in [0,1]$ such that under the tent map, $x_m = L$ implies $T^m(\alpha) \in [0, \frac{1}{2}]$ and $x_m = R$ implies $T^m(\alpha) \in (\frac{1}{2}, 1]$. Furthermore, two different initial points cannot have the same itinerary, i.e. for $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$, the itineraries of $\alpha$ and $\beta$ under the tent map will be different.

**Example 2** Consider $\xi = \frac{1}{4}$ and $X = (\xi, T(\xi), T^2(\xi), T^3(\xi), T^4(\xi), T^5(\xi)...)$.

Then,

$$\frac{1}{4} \mapsto L$$

$$T(\tfrac{1}{4}) = \tfrac{1}{2} \mapsto L$$

$$T^2(\tfrac{1}{4}) = 1 \mapsto R$$

$$T^3(\tfrac{1}{4}) = 0 \mapsto L$$

$$T^4(\tfrac{1}{4}) = 0 \mapsto L$$

$$T^5(\tfrac{1}{4}) = 0 \mapsto L$$

$$T^6(\tfrac{1}{4}) = 0 \mapsto L$$

$$\vdots$$

$$T^n(\tfrac{1}{4}) = 0 \mapsto L$$

So the *itinerary* of $\xi$ under the tent map is $X = (L\ L\ R\ L\ L\ L\ L\ L\ . . .)$

Suppose now we want to modify our definition and example. Instead of looking at iterations of the tent map $T(x)$, lets look at iterations of the tent map composed with itself $T_2(x)$. And looking at the itinerary of a point $x_0 \in [0,1]$, assign L to the $n$th point of the orbit if $T_2^n(x_0) \in [0, \frac{1}{4}]$ and assign R to the $n$th point of the orbit if $T_2^n(x_0) \in (\frac{1}{4}, 1]$. Our next goal is to take an arbitrary point in $[0,1]$ and find its itinerary under $T_2(x)$. Then we will find what point in $[0,1]$ is mapped to that itinerary under the normal tent map $T(x)$. This process will create our topological semi-conjugacy $h(x)$ between $T_2(x)$ and $T(x)$. Of course we will have to do this numerically so there will be some small error in our function.

**Example 3** Take $x_0 = \frac{1}{4}$ and its itinerary under $T_2(x)$ is $X = (L\ R\ L\ L\ L\ L\ L\ L\ L\ \ldots)$. We want to find what point $y_0 \in [0,1]$ has that same itinerary under the normal tent map. Since the first letter in our itinerary $X$ is L we know that $y_0$ must be in $[0, \frac{1}{2}]$. The next letter in our itinerary $X$ is R so we know that $y_0$ must actually be in $[\frac{1}{4}, \frac{1}{2}]$. We can keep repeating this process iteratively until we get a small enough interval to where we can make a good approximation of $y_0$. Continuing this example we get:

L $\mapsto [0, \frac{1}{2}]$

R $\mapsto [\frac{1}{4}, \frac{1}{2}]$

L $\mapsto [\frac{3}{8}, \frac{1}{2}]$

L $\mapsto [\frac{7}{16}, \frac{1}{2}]$

L $\mapsto [\frac{15}{32}, \frac{1}{2}]$

L $\mapsto [\frac{31}{64}, \frac{1}{2}]$

L $\mapsto [\frac{63}{128}, \frac{1}{2}]$

L $\mapsto [\frac{127}{256}, \frac{1}{2}]$

L $\mapsto [\frac{255}{512}, \frac{1}{2}]$

L $\mapsto [\frac{511}{1024}, \frac{1}{2}]$

At this point we have a fairly good approximation and it is clear that the limiting value of $y_0 = \frac{1}{2}$. So $h(\frac{1}{4}) = \frac{1}{2}$. Thus we have created the first point of our topological semi-congjugacy between $T_2(x)$ and $T(x)$.

# 2 Semi-Conjugacy using MATLAB

We know that $h(\frac{1}{4}) = \frac{1}{2}$, thus we will continue this process for 10,000 other points in $[0,1]$ in order to get a very good approximation of what $h(x)$ looks like. First we will break up the interval $[0,1]$ into 10,000 evenly spaced points. Next we generate the itineraries of each of these points in $T_2(x)$. Finally, we will look at each individual itinerary we generated and determine what point under the regular tent map is mapped to that itinerary, using the same process we did in **Example 3**. We can generate the following code that will complete all three steps at once with the final output being a graph of the semi-conjugacy $h(x)$ :

```
syms y(x)
y(x) = piecewise(0<=x<=.25, 4*x, .25<x<=.5, ...
    2-4*x, .5<x<=.75, 4*x-2,.75<x<=1,4-4*x);
p = linspace(0,1,1001);
u = [];
v = [];
j=1;
digits(64);
while j < 1002
    r = p(j);
    a = 1:10;
    i=1;
    while i < 11
       if r<=.25
           a(i) = 1;
       elseif r > .25
           a(i) = -1;
       end
     r = y(r);
        i=i+1;
    end
    m=0;
    n=1;
    b = [m,n];
    c = zeros(10,0);
    k=1;
    while k < 11
        c(k) = a(k);
        if prod(c) > 0
            if c(k) == 1
                n=((m+n)/2);
                 b = [m,n];

            elseif c(k) == -1
                n=((m+n)/2);
                 b = [m,n];
            end
        elseif prod(c)<0
            if c(k) == 1
                m=((m+n)/2);
                b = [m,n];
            elseif c(k) == -1
                m=((m+n)/2);
                 b = [m,n];
            end
        end
       k = k+1;
    end
    d=(m+n)/2;
    u(j) = p(j);
    v(j) = d;
    j=j+1;
end
plot(u,v);
```

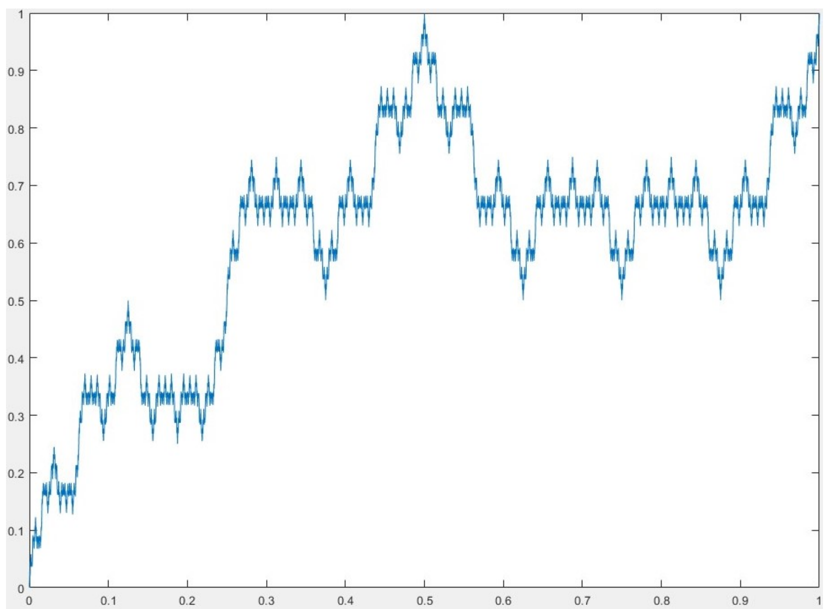
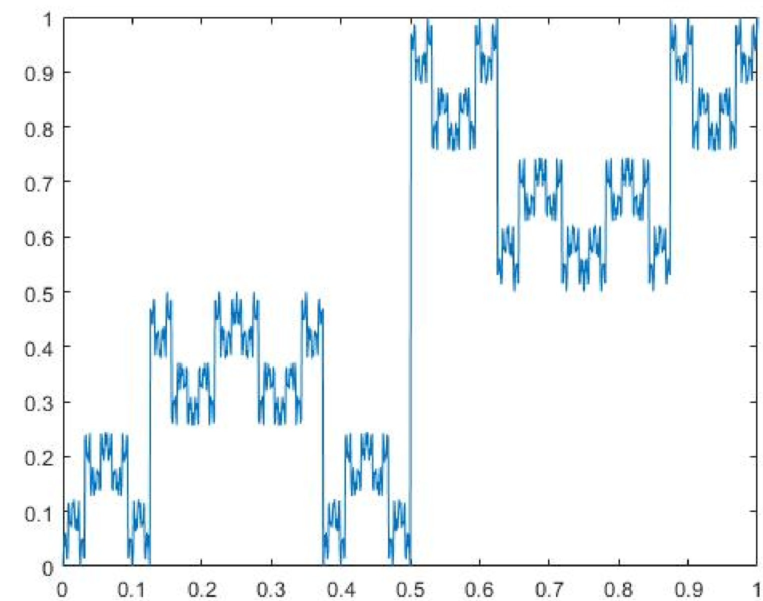



Figure 1: MATLAB code and graph of $h(x)$

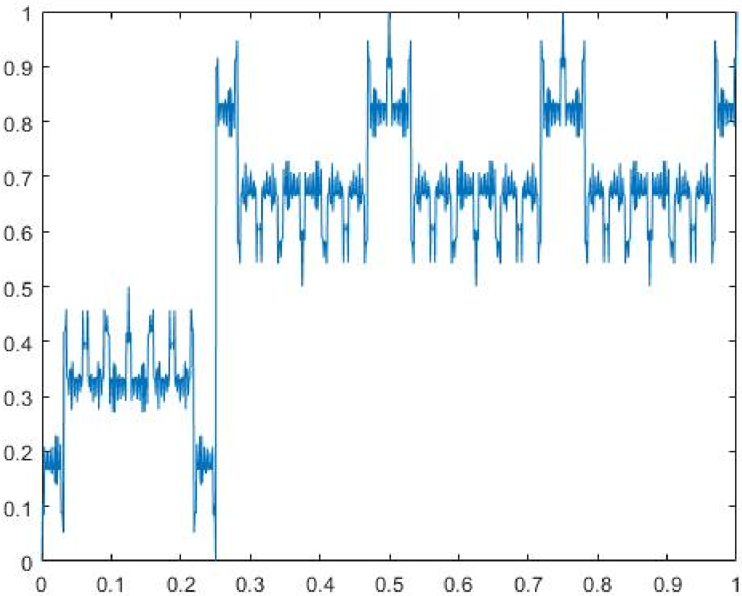
We can now see that the semi-conjugacy created between $T_2(x)$ and the tent map, $h(x)$, is actually a **fractal:** a nowhere differentiable function that displays self-similarity.
We can play around with the MATLAB code and create more semi-conjugacies. We will create another semi-conjugacy from $T_2(x)$ to the tent map, but now suppose we assign L to the $n$th point of the orbit if $T_2^n(x_0) \in [0, \frac{1}{2}]$ and assign R to the $n$th point of the orbit if $T_2^n(x_0) \in (\frac{1}{2}, 1]$. After modifying the code, we get the following semi-conjugacy graph:

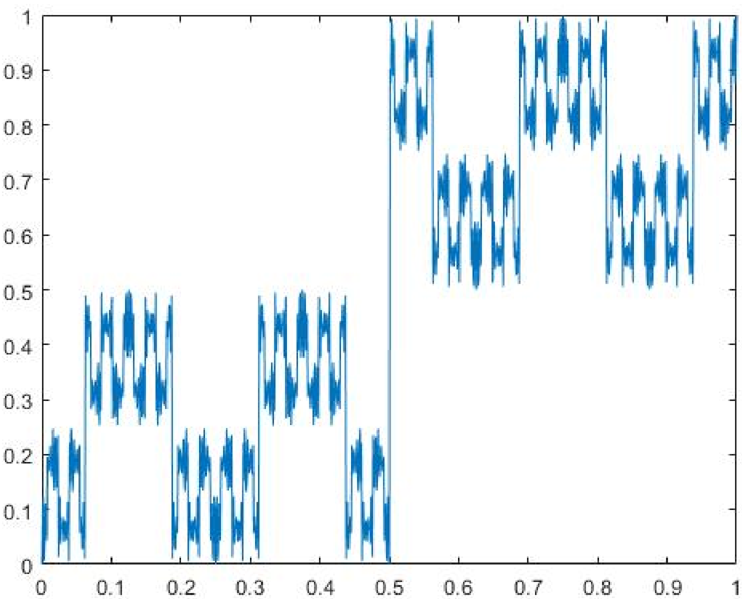



Recall that $T_2(x)$ was just the tent map composed with itself once and we then created semi-conjugacies from $T_2(x)$ to the tent map. Now lets try and create more semi-conjugacies, but this time from the tent map composed with itself two or three times instead of just once.

We have that $T \circ T \circ T(x) = T_3(x) = \begin{cases} 8x & 0 \leq x \leq \frac{1}{8} \\ 2 - 8x & \frac{1}{8} \leq x \leq \frac{2}{8} \\ 8x - 2 & \frac{2}{8} \leq x \leq \frac{3}{8} \\ 4 - 8x & \frac{3}{8} \leq x \leq \frac{4}{8} \\ 8x - 4 & \frac{4}{8} \leq x \leq \frac{5}{8} \\ 6 - 8x & \frac{5}{8} \leq x \leq \frac{6}{8} \\ 8x - 6 & \frac{6}{8} \leq x \leq \frac{7}{8} \\ 8 - 8x & \frac{7}{8} \leq x \leq 1 \end{cases}$

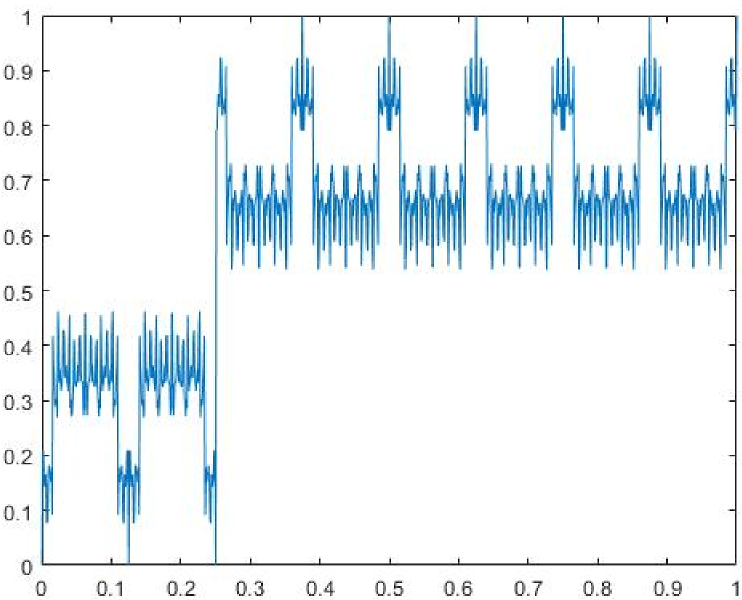
Using the same MATLAB code we can create two semi-conjugacies from $T_3(x)$ to the tent map:

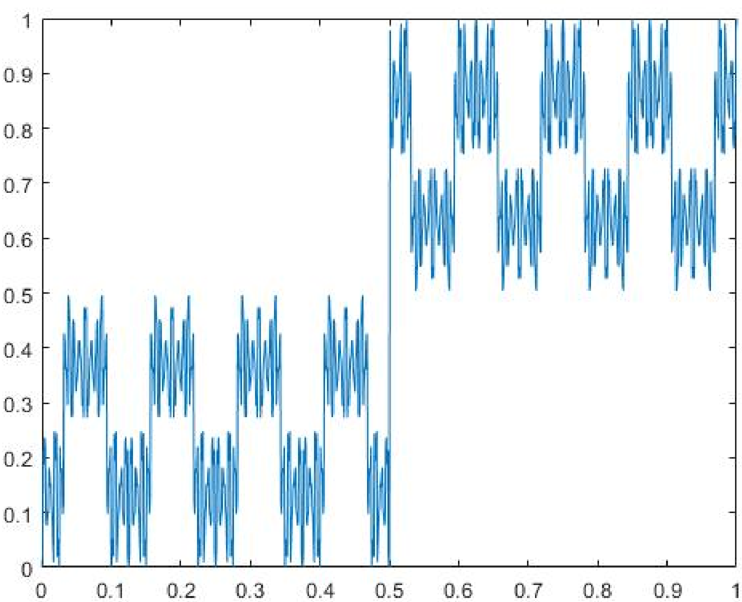



The first graph being the semi-conjugacy when we assign L to the $n$th point of the orbit if $T_3^n(x_0) \in [0, \frac{1}{4}]$ and assign R to the $n$th point of the orbit if $T_3^n(x_0) \in (\frac{1}{4}, 1]$. The second being when we assign L to the $n$th point of the orbit if $T_3^n(x_0) \in [0, \frac{1}{2}]$ and assign R to the $n$th point of the orbit if $T_3^n(x_0) \in (\frac{1}{2}, 1]$.

Now we will create two semi-conjugacies from $T_4(x)$ to the tent map. We have:




The first graph being the semi-conjugacy when we assign L to the $n$th point of the orbit if $T_4^n(x_0) \in [0, \frac{1}{4}]$ and assign R to the $n$th point of the orbit if $T_4^n(x_0) \in (\frac{1}{4}, 1]$. The second being when we assign L to the $n$th point of the orbit if $T_4^n(x_0) \in [0, \frac{1}{2}]$ and assign R to the $n$th point of the orbit if $T_4^n(x_0) \in (\frac{1}{2}, 1]$.

# 3 Conclusion

After viewing the graphs of a few topological conjugacies, it is clear why it is so difficult to find these conjugacies between functions. Many topological conjugacies create beautiful chaotic graphs, in addition, the homeomorphism of systems shows that the systems have the same dynamical structure. Conjugacy of systems was popularized in the 1900's as the study of topological dynamics developed. George Birkhoff is considered the founder of the field, but he based much of his work on the work of Poincare, Bendixion, Ansov and others. A still developing field that is sure to provide many useful breakthroughs in the study of dynamical systems.

# 4 References


- Ledis, Dennis. *Semi-Conjugacies to tent maps, transitivity and patterns, and inverse limit spaces* 2011. University of Florida, PhD dissertation.
- Alligood, Kathleen T., et al. *Chaos: an Introduction to Dynamical Systems*. Springer, 2010.
- Strogatz, Steven H. *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*. CRC Press, 2019.